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KK-masses in dipole deformed field theories

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Abstract: We reconsider aspects of non-commutative dipole deformations of field theories. Among our findings there are hints to new phases with spontaneous breaking of translation invariance (stripe phases), similar to what happens in Moyal-deformed field theories. Furthermore, using zeta-function regularization, we calculate quantum corrections to KK-state masses. The corrections coming from non-planar diagrams show interesting but non-universal behaviour. Depending on the type of interaction the corrections can make the KK-states very heavy but also very light or even tachyonic. Finally we point out that the dipole deformation of QED is not renormalizable!

Keywords: AdS-CFT Correspondence, Field Theories in Higher Dimensions, Non-Commutative Geometry.

Contents

1. Introduction

In recent years, a huge amount of work has been devoted to the study of non-commutative field theories. The main focus has been the Moyal bracket case, based on a non-trivial commutator of coordinates $[x^\mu, x^\nu] = i\Theta^{\mu\nu}$. Part of the interest in these theories is triggered by the observation that they are realized in string theory on the world volume of D-branes in a B-field background. The B-field is an antisymmetric second rank tensor, and the Moyal bracket deformation is obtained by choosing a polarization of the B-field with both indices along the directions of the world volume of the D-brane. It is also possible to arrange the B-field in a different way, with one index along the brane directions and the other one transverse to them. Also in this case one obtains a deformation of the world volume theory. The deformation in question is defined by the star-product

$$\begin{aligned} \phi_1(x) \star \phi_2(x) &:= \exp\Big[-\frac{1}{2}(L_2^\mu \partial_\mu^x - L_1^\mu \partial_\mu^y)\Big] \phi_1(x)\, \phi_2(y)\Big|_{x=y} \\ &= \phi_1\left(x - \frac{L_2}{2}\right) \phi_2\left(x + \frac{L_1}{2}\right) \ , \end{aligned} \tag{1.1}$$

which was first constructed in [1] by considering T-duality of Moyal-bracket deformed theories. Its basic field theoretical properties have been first studied in [2]. As explained there, $L^\mu_{1,2}$ are the so-called dipole lengths of the fields ϕ_1 and ϕ_2. Associativity of this

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star-product

$$(\phi_i \star \phi_j) \star \phi_k = \phi_i \star (\phi_j \star \phi_k) \ , \tag{1.2}$$

demands that the dipole length of a product of fields be the sum of the dipole length of each field

$$L_{\phi_1 \star \cdots \star \phi_n} = L_{\phi_1} + \ldots + L_{\phi_n} \ , \tag{1.3}$$

i.e. the dipole length is *additive*. Supergravity backgrounds dual to dipole deformed field theories have been investigated in [3 − 7] whereas various field theory aspects have been described in [8 − 10]. It is important to note that these dipole lengths are always related to U(1) symmetries of the (undeformed) field theory under consideration. Without recourse to string theory, a dipole deformation of a field theory can be defined by introducing the dipole lengths of the fields according to

$$L^\mu_\phi = \ell^\mu_a Q^a_\phi \ , \tag{1.4}$$

where Q^a_ϕ are U(1) charges of the field ϕ and the matrix ℓ^μ_a picks out a certain linear combination. The Lagrangian of the deformed field theory is then obtained by simply multiplying the fields with the star-product (1.1). Since the terms in the Lagrangian are neutral under the chosen U(1) symmetries, it is clear that the dipole lengths of all the fields in a Lagrangian term add up to zero. In this case one is allowed to delete one star from the product under the integral, i.e.

$$\int \mathrm{d}x\ \phi_1(x) \star \phi_2(x) \star \cdots \star \phi_n(x) = \int \mathrm{d}x\ \phi_1(x)\, \phi_2(x) \star \cdots \star \phi_n(x) \ . \tag{1.5}$$

An immediate consequence is that the quadratic terms in the action remain undeformed, and therefore the propagators in the deformed quantum theory are the same as those in the undeformed one. The integral defines the trace on the C^*-algebra of functions defined by the deformed product. In the case of matrix valued fields this also includes a trace on the matrix indices. However, the necessary cyclicity condition is only fulfilled if the total dipole length of the integrand is *zero*, because only then one can delete one star under the integral and therefore permute the fields cyclically.

Since the dipole deformation is based on the presence of U(1) charges, it is clear that the dipole moment of neutral fields is zero. For example, there is no non-trivial dipole deformation of real scalar field theory and for the same reason the dipole length of gauge fields has to vanish. In the case of complex fields, demanding that

$$(\phi^\dagger \star \phi)^\dagger = \phi^\dagger \star \phi \ , \tag{1.6}$$

shows that $L_{\phi^\dagger} = -L_\phi$.

Although this structure is very similar to the well-known Moyal bracket deformation of field theories, it has been relatively little investigated. However, recently a very interesting proposal has been put forward by Núñez and Gürsoy [11]. They considered the supergravity background as a specific dipole deformation of the theory living on D5-branes wrapped on an S^2 inside a Calabi–Yau, in such a way as to preserve $\mathcal{N} = 1$ supersymmetry in the four dimensional flat part of the world volume. Such supergravity duals of confining gauge theories are in general plagued by a rather unwelcome feature: the scale of the masses of the KK-states coming from the compactified part of the world volume is of the same order as the scale of the four dimensional gauge theory of interest. Therefore, one cannot disentangle the interesting strongly coupled gauge theory dynamics from the artefacts of these KK-states.

Núñez and Gürsoy pointed out that this situation might be improved if one considers a dipole deformed D5-brane theory. More specifically, using the techniques developed in [12] they constructed the supergravity background of such a theory in a B-field background with one index of the B-field along the S^2 and another one transverse to the D5-brane volume. They noted then that the volume of the compact internal manifolds in the deformed background are smaller than in the undeformed one, therefore indicating a possible disentanglement of the KK-states from the interesting gauge theory dynamics. Further aspects of KK-states in these supergravity backgrounds have been discussed in [13, 14].

This work motivated us to investigate the issue of KK-state masses in dipole deformed theories from the purely field theoretical point of view. Of course, the underlying theory [11] is very complicated, and since its UV completion ultimately is little string theory, it is not really a field theory. We will therefore study much simpler examples of dipole deformations of field theories compactified on a circle. We found however that even the uncompactified theories show a rather interesting behaviour, that so far has not been discussed in the literature. Therefore, we will start in section 2 by investigating dispersion relations in dipole deformed scalar field theories in three, four and five dimensions. We will find indications for a phase transition at a certain critical dipole length in three dimensions and five dimensions, whereas the four dimensional theory is free of such behaviour at least in the weak coupling regime.

In section 2 we move on to study the same scalar field theory compactified on a circle. We find that the two-point amplitudes can be expressed in terms of Epstein zeta functions. We will phrase the discussion in the language of thermal field theory by introducing the inverse circumference $T = 1/(2\pi R)$. Interestingly, the amplitudes depend on the dipole length only through the non-integer part of the product TL and, for TL being close to an integer, the non-planar two-point amplitudes grow without bound. Depending then on the type of interactions the non-planar corrections to the KK-masses can be very large and of positive or negative sign. In the latter case this indicates a tachyonic instability (this is very similar to the behaviour of non-commutative Moyal bracket deformed theories compactified on a non-commutative torus [15]).

In section 3 we investigate the dipole deformation of the massless Wess–Zumino model. The undeformed model is supersymmetric but, since the U(1) symmetry that we use to define the dipole moments is the $\mathcal{R}$-symmetry, supersymmetry is broken in the deformed theory. We show that the leading divergences appearing in the planar graphs still cancel whereas the corresponding (leading) L dependence of the non-planar amplitudes do not cancel.

In section 4 we have a very brief look to dipole deformed QED. We use the U(1) gauge symmetry for the dipole deformation. It turns out that already the one-loop corrections to the polarization tensor give rise to momentum dependent divergences that can not be absorbed in a controlled way in the parameters of the tree level Lagrangian, therefore the model is not renormalizable!

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We conclude with a summarizing discussion in section 5. In the appendix we outline the analytic continuation of the Epstein zeta function and provide some details of the calculations of section 3.

2. The bosonic dipole theory

Following the preceding section, we begin by formulating a dipole theory for complex scalar fields ϕ and $\phi^\dagger$ with quartic interactions. The action is

$$S = \int \mathrm{d}^D x \, \Big(\partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi - \lambda(\phi^\dagger \star \phi \star \phi^\dagger \star \phi) - g(\phi^\dagger \star \phi^\dagger \star \phi \star \phi) \Big)(x) \ . \tag{2.1}$$

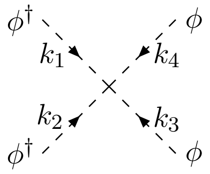
As said, we have deleted the star-product in the quadratic terms. The bare propagator is thus as in the commutative theory

$$\widetilde{\Gamma}^{(2)}_{\phi\phi^\dagger,\mathrm{bare}}(k) = \frac{i}{\boldsymbol{k}^2 - m^2 + i\varepsilon} \ . \tag{2.2}$$

There are two possible orderings of the fields in the interaction vertex. In the first one with coupling λ, the star-product shifts the arguments of all fields in the same direction and by the same amount. Therefore it can be undone by a compensating shift of the integration variable. Thus, the λ-coupling gives the same interactions as in the undeformed theory. Only the second term with coupling g produces a new form of the interaction. In momentum space it is

$$\begin{aligned} S_g &= -g \int \mathrm{d}^D x \ (\phi^\dagger \star \phi^\dagger \star \phi \star \phi)(x) = \\ &= -g\,(2\pi)^D \int \left(\prod_{i=1}^{4} \frac{\mathrm{d}^D \boldsymbol{k}_i}{(2\pi)^D} \right) \delta^D(\Sigma \boldsymbol{k}_i) \times \\ &\qquad \times \mathrm{e}^{-iL_\phi \cdot (\boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3 + \boldsymbol{k}_4)/2} \ \widetilde{\phi}^\dagger(\boldsymbol{k}_1)\, \widetilde{\phi}^\dagger(\boldsymbol{k}_2)\, \widetilde{\phi}(\boldsymbol{k}_3)\, \widetilde{\phi}(\boldsymbol{k}_4) \ , \end{aligned} \tag{2.3}$$

and so one obtains the vertex for the g-term in Minkowski space $\mathbb{R}^{1,D-1}$



$$= -ig \, \exp\left(-\frac{i}{2} L_\phi \cdot (\boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3 + \boldsymbol{k}_4) \right) \ . \tag{2.4}$$

We also note that in the case $g = -\lambda$ the interaction is given by the square of the $\star$-product commutator $[\phi^\dagger, \phi]$.

2.1 The uncompactified theory

As shown in figure 1, the deformed vertex allows for two inequivalent one-loop corrections to the two-point function. There is a planar graph in which all the phases cancel, and a non-planar one in which the phase depends on the loop momentum $\boldsymbol{k}$ as well as on the external momentum $\boldsymbol{p}$. Since the planar amplitude is independent of the external

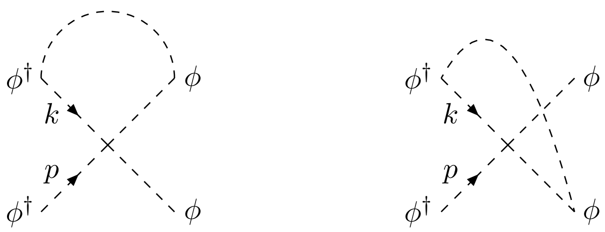


Figure 1: The deformed vertex allows for a planar and a non-planar one-loop correction to the two-point function.

momentum we can absorb it in an infinite renormalization of the mass or set it to zero by using dimensional regularization. The non-planar amplitude is given by

$$-i\mathcal{A}_{\text{n-p}} = 2g\cos(\boldsymbol{p}\cdot L)\int \frac{\mathrm{d}^D\boldsymbol{k}}{(2\pi)^D}\,\frac{\mathrm{e}^{i\boldsymbol{k}\cdot L}}{\boldsymbol{k}^2 - m^2 + i\epsilon}\,. \tag{2.5}$$

The amplitude can be easily evaluated by Wick rotating to Euclidean signature and introducing a Schwinger parametrization

$$\mathcal{A}_{\text{n-p}} = 2g\cos(\boldsymbol{p}_E\cdot L_E)\int \mathrm{d}\alpha\int\frac{\mathrm{d}^D\boldsymbol{k}_E}{(2\pi)^D}\,\exp\Big(-\alpha(k_E^2+m^2)+i\boldsymbol{k}_E\cdot L_E\Big) = \tag{2.6}$$

$$= 2g\cos(\boldsymbol{p}_E\cdot L_E)\int \mathrm{d}\alpha\left(\frac{1}{4\pi\alpha}\right)^{D/2}\exp\left(-\alpha m^2 - \frac{L^2}{4\alpha}\right) = \tag{2.7}$$

$$= 2g\cos(\boldsymbol{p}_E\cdot L_E)\,\frac{m^{D-2}}{\sqrt{(2\pi)^D}}\,\frac{K_{D/2-1}(|Lm|)}{|Lm|^{(D-2)/2}}\,, \tag{2.8}$$

where $K_n(x)$ is a modified Bessel function of the second kind. The UV divergence is regulated in the non-planar graph by the dipole length. The leading behaviour for $Lm \to 0$ is

$$\mathcal{A}_{\text{n-p}} = \frac{g}{2\pi^{D/2}}\,\cos(\boldsymbol{p}_E\cdot L_E)\,\Gamma\left(\frac{D-2}{2}\right)L^{2-D}\,. \tag{2.9}$$

This gives rise to a modified dispersion relation of the form

$$E^2 = p^2 + \frac{g}{2\pi^{D/2}}\,\cos(pL)\,\Gamma\left(\frac{D-2}{2}\right)L^{2-D}\,, \tag{2.10}$$

where for simplicity we took the massless limit and also considered the $D-1$ momentum $\vec{p}$ to be parallel to L. This in particular implies that L is a spacelike vector and that there is a coordinate system in which L has non-vanishing component only in the D-th direction $L_\mu = (0, 0, \ldots, L)$.

Due to the presence of the cosine, the correction term can be negative for some ranges of momentum. It is natural then to ask if the dispersion relation can develop a minimum away from the origin in momentum space. We can view this as a condition on L and define a first critical dipole length as

$$\frac{\partial E^2}{\partial p} = 0 \;\to\; \left[\frac{4\pi^{D/2}}{g\Gamma(D/2-1)}L_{c1}^{D-4}\right]\cdot(pL) = \sin(pL)\,, \tag{2.11}$$

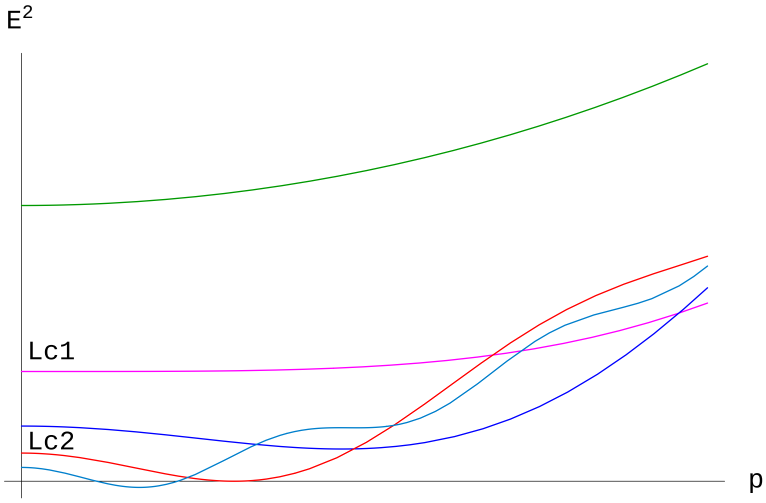


Figure 2: Dispersion relation for different dipole lengths in $D = 3$. At L_{c1} it develops a minimum away from $p = 0$ and at L_{c2} it touches $E = 0$.

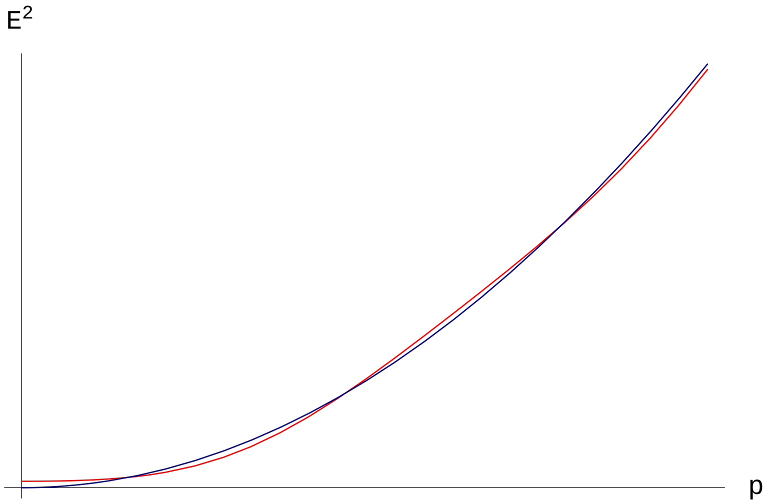


Figure 3: Dispersion relation for different dipole lengths in $D = 4$, only small wiggling around $E^2 = p^2$ is observed.

$$\Longrightarrow L_{c1}^{D-4} = \frac{g\Gamma(D/2-1)}{4\pi^{D/2}}\,. \tag{2.12}$$

Note that at weak coupling this can not be fulfilled in $D = 4$! In $D = 3$ it is fulfilled for relatively large L. In the massive theory this would still be approximately valid if we assume the mass to be small such that Lm is small, and the right hand side of the dispersion relation is modified by the addition of the mass term. In $D = 5$ this condition is fulfilled for small L. The five-dimensional theory however is bound to inherit the divergences of the undeformed theory in the planar sector, and therefore will not be renormalizable. We can also define a second critical length L_{c2} to be the value where the right hand side of (2.10) becomes negative. This would mean that some of the modes have imaginary energies and therefore show a tachyonic instability. Again, in the four-dimensional theory this can not

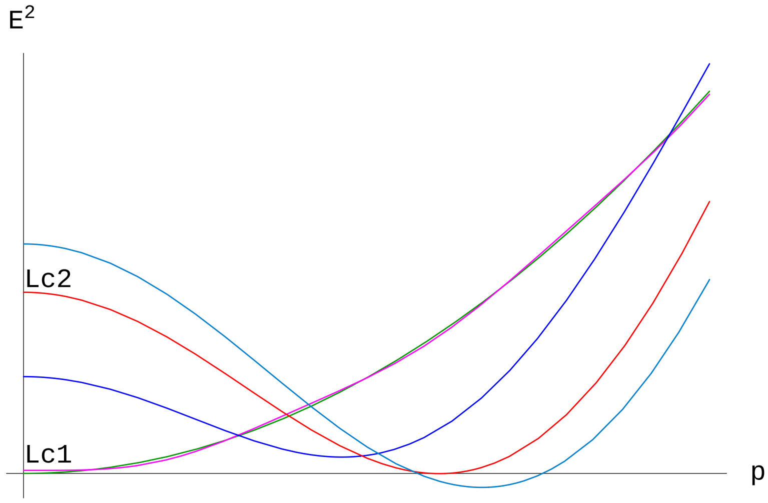


Figure 4: Dispersion relation for different dipole lengths in $D = 5$. At L_{c1} it develops a minimum away from $p = 0$ and at L_{c2} it touches $E = 0$. The effect is more pronounced for smaller L.

happen at weak coupling. In $D = 3$ this happens however for $L \geq L_{c2} \approx \frac{49}{g}$, whereas in $D = 5$ it happens for $L \leq L_{c2} \approx \frac{g}{308}$. This would imply that the mode $p_{\min}$ that minimizes (2.10) develops a vacuum expectation value. Since it is a non-zero momentum mode that condenses the new ground state spontaneously breaks translation invariance in the direction of the dipole moment! This behaviour is reminiscent of the behaviour of Moyal deformed ϕ^4 theory where it was argued in [16] that the UV/IR mixing gives rise to stripe phases upon adding sufficiently negative tree level mass terms. This was later on confirmed by lattice studies in [17]. In our case of the dipole deformed theory, we expect the results to be qualitatively valid in the presence of a small positive or negative mass squared term at tree level. It would be rather interesting to see if these results can be confirmed by a lattice study of the theory in $D = 3$.[1]

Four-point function. Let us have now a quick look to the quantum corrections to the four-point function. We specialize to $D = 4$ and study the possible divergences. The one-loop correction to the four-point function can be computed from the possible Wick contractions of

$$\Gamma^{(4)}_{1-\text{loop}}(x_1, x_2, x_3, x_4) = \int \mathrm{d}^4 z\, \mathrm{d}^4 w \left\langle \phi^\dagger(x_1)\phi^\dagger(x_2)\phi(x_3)\phi(x_4) \times \right. \tag{2.13}$$
$$\times \left[\lambda(\phi^\dagger\phi)^2(w) + g\left(\phi^\dagger\phi\right)\left(w + \frac{L}{2}\right)(\phi^\dagger\phi)\left(w - \frac{L}{2}\right)\right] \times$$
$$\left. \times \left[\lambda(\phi^\dagger\phi)^2(z) + g\left(\phi^\dagger\phi\right)\left(z + \frac{L}{2}\right)(\phi^\dagger\phi)\left(z - \frac{L}{2}\right)\right]\right\rangle .$$

[1] We would like to note that in $D = 3$ the coupling constant has mass dimension one and it might be that the relevant dimensionless expansion parameter is Lg. In the latter case the condition for appearance of tachyonic modes could still turn out to be at strong effective coupling Lg.

The divergences in spacetime arise from the pointwise multiplication of two propagators

$$\langle \phi(x)\phi^\dagger(y)\rangle = \Delta(x-y) \text{ and } \Delta(z)\Delta(z) \approx \log(\Lambda)\delta(z)\ . \tag{2.14}$$

The non-planar contributions amount to multiplication of two propagators with arguments shifted by multiples of $L/2$, e.g. $\Delta(w-z+L/2)\,\Delta(w-z-L/2)$. Performing all possible Wick contractions and retaining only the divergent contributions we find

$$\begin{aligned}\Gamma^{(4)}_{1-\text{loop,div}} = \frac{-1}{8\pi^2}\log(\Lambda)\int \mathrm{d}^4 z\, \Big\langle \phi^\dagger(x_1)\phi^\dagger(x_2)\phi(x_3)\phi(x_4)\times \\ \times\Big[(20\lambda^2+2g^2)(\phi^\dagger\phi)^2(z) + (16\lambda g + 4g^2)(\phi^\dagger\phi)\left(z+\frac{L}{2}\right)(\phi^\dagger\phi)\left(z-\frac{L}{2}\right) + \\ +2g^2(\phi^\dagger\phi)(z+L)(\phi^\dagger\phi)(z-L)\Big]\Big\rangle\ . \end{aligned}\tag{2.15}$$

This shows that the theory with the λ and g interactions in $D=4$ is strictly speaking not renormalizable. The additional divergence is however proportional to a dipole star-product term with twice the dipole length. The divergence can therefore be absorbed into a new tree level term in the action with star-product and dipole length $2L$. Since this term arises at one loop it is also natural to assume that its coupling constant is proportional to g^2; its non-planar one-loop contribution to the two-point function can therefore be neglected to order g! It is clear, that at order g^3 a new vertex with star-product structure and dipole length $3L$ will be induced. In general, at order g^n one needs to assume a tree level Lagrangian of the form

$$\mathcal{L} = \sum_{k=0}^{n} g_k\,(\phi^\dagger\phi)\Big(x-k\frac{L}{2}\Big)\,(\phi^\dagger\phi)\Big(x+k\frac{L}{2}\Big)\ , \tag{2.16}$$

where the couplings can be assumed to obey $g_k \approx g_0^k$ for $k>0$, i.e. $g_1=\mathcal{O}(g_0)$, $g_2=\mathcal{O}(g_0^2)$, etc. Although the theory is not renormalizable in the usual sense, the divergences stemming from the just discussed corrections to the four-point function are under control, and can be absorbed in counterterms of star-product form. This observation has been made already [2].

2.2 Corrections to KK-masses

Let us now consider the dipole scalar field theory on $\mathbb{R}^{1,D-2}\times\mathbf{S}^1$. The periodicity on the $\mathbf{S}^1$ direction, with radius R, implies that momenta along that direction are discrete. Since we deal with a complex scalar field with a U(1) global symmetry, the field can pick up a phase once transported around the circle. Thus, if we impose twisted boundary conditions

$$\phi(x+2\pi R) = \mathrm{e}^{2\pi i\alpha}\phi(x)\ ,\quad \alpha\in[0,1)\ , \tag{2.17}$$

the momentum along the $\mathbf{S}^1$ is

$$p_{\mathbf{S}^1} = \frac{n+\alpha}{R}\ . \tag{2.18}$$

From now on, when studying the mass corrections to the KK-states of the tower coming from the $\mathbf{S}^1$, we will put the bare mass to zero, $m=0$. We will also consider the dipole length to be oriented only along the $\mathbf{S}^1$.

After an appropriate Wick rotation (we assume time to be one of the non-compact dimensions) the non-planar amplitude takes now the form

$$\mathcal{A}_{\rm n-p} = 2gT \int \frac{\mathrm{d}^d k_E}{(2\pi)^d} \sum_{n\in\mathbb{Z}} \cos\Big(2\pi TL(n_p - n)\Big) \frac{1}{k_E^2 + [2\pi T(n+\alpha)]^2} \,. \tag{2.19}$$

As already stated in the introduction, we used the language of finite field theory and have set $T = \frac{1}{2\pi R}$. The external momentum is given by $2\pi T n_p$. The amplitude is of course independent of the inflowing momentum along the non-compact directions. Note that the argument of the cosine is independent of the twist parameter α.

We notice that $TL \equiv b$ is reduced to the lattice $\left[-\frac{1}{2}, \frac{1}{2}\right)$, because of the periodicity of the cosine function. Therefore, the amplitude does not care about the magnitude of TL (which can be adjusted by changing L), but only about its non-integer part. Now we split the cosine into exponentials and further perform the k-integration by using the Schwinger parametrization. This yields

$$\begin{aligned} \mathcal{A}_{\rm n-p} &= g\,\frac{T^{d-1}}{4\pi}\,\mathrm{e}^{2\pi i b n_p} \sum_{n\in\mathbb{Z}} \int_0^\infty \mathrm{d}t\; t^{-d/2}\; \mathrm{e}^{-\pi t(n+\alpha)^2 - 2\pi i b n} \;+ \text{c.c.} = \\ &= g\,\frac{T^{d-1}}{4\pi}\,\frac{\Gamma\left(1-\frac{d}{2}\right)}{\pi^{(2-d)/2}} \left\{ \mathrm{e}^{2\pi i b n_p} \sum_{n\in\mathbb{Z}} \frac{\mathrm{e}^{-2\pi i n b}}{|n+\alpha|^{2-d}} + \mathrm{e}^{-2\pi i b n_p} \sum_{n\in\mathbb{Z}} \frac{\mathrm{e}^{2\pi i n b}}{|n+\alpha|^{2-d}} \right\} \,. \end{aligned} \tag{2.20}$$

We can now evaluate the amplitude using zeta-function techniques. As explained in the appendix the sums in the amplitude give a representation of the one-dimensional Epstein zeta function for $2 - d > 1$. The amplitude can be regularized by analytic continuation of the Epstein zeta function. We also make use of the functional identity (A.10) of the appendix and can write the result as

$$\mathcal{A}_{\rm n-p} = g\,\frac{T^{d-1}}{2\pi}\,\frac{\Gamma\left(\frac{d-1}{2}\right)}{\pi^{(d-1)/2}}\,\mathrm{Re}\left[\mathrm{e}^{2\pi i b(n_p+\alpha)}\; Z\left|\begin{matrix} b \\ \alpha \end{matrix}\right|(d-1)\right] \,. \tag{2.21}$$

Let us analyse some of the properties of this amplitude. We begin with its singularity structure. From (A.8) we see that there are two possible pole-like singularities

$$\mathcal{A}_{\rm n-p} = g\,\frac{T^{d-1}}{2\pi}\cos\Big(2\pi b(n_p+\alpha)\Big)\left(\frac{2\delta_{\alpha,0}}{d-2} - \frac{2\delta_{b,0}}{d-1} + \text{two regular terms}\right) \,. \tag{2.22}$$

The physical meaning of these two poles is the following:

- for $d = 2$ and $\alpha = 0$ the pole corresponds to the infrared divergence of the dimensionally reduced theory in two dimensions,
- for $d = 1$ and $b = 0$ the zeta function only regularizes the infrared divergence of the theory in ($D = 2$), but leaves us with the ultraviolet one.

The presence of the poles can therefore be traced back to the simultaneous presence of UV and IR divergences.

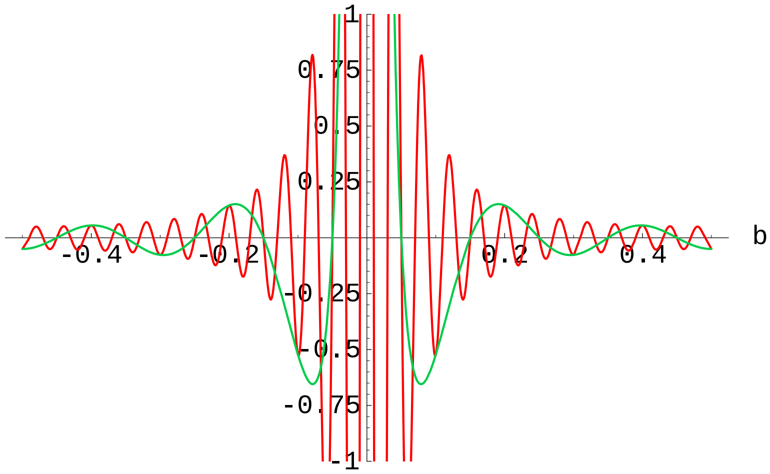


Figure 5: Non-planar amplitude as a function of the parameter b in $d = 3$ and without twist. The amplitude is plotted for mode number $n = 3$ and mode number $n = 10$. Although the amplitude is finite at $b = 0$ it grows without bounds for $b \to 0$!

When the twist parameter α vanishes, the amplitude is factorized as

$$\mathcal{A}_{\rm n-p} = g\,\frac{T^{d-1}}{2\pi}\cos(2\pi b n_p)\left\{\frac{2}{d-2} - \frac{2\delta_{b,0}}{d-1} + \right. \\ \left. + \int_1^\infty {\rm d}t \left[t^{-d/2}\left(\vartheta_3(b|it) - 1\right) + t^{(d-3)/2}\,{\rm e}^{-\pi b^2 t}\left(\vartheta_3(ibt|it) - \delta_{b,0}\right)\right]\right\} . \tag{2.23}$$

If the parameter b vanishes it can be written in terms of generalized Riemann zeta functions as

$$\mathcal{A}_{\rm p} = g\frac{T^{d-1}}{2\pi}\,\frac{\Gamma\left(1-\frac{d}{2}\right)}{\pi^{(1-d/2)}}\left[\zeta_R(2-d,\alpha) + \zeta_R(2-d,1-\alpha)\right] , \tag{2.24}$$

which is of course the zeta-function regularized result for the planar amplitude. For example in $d = 3$ and without twist we get

$$\mathcal{A}_{\rm p} = \frac{(2\lambda + g)T^2}{6} , \tag{2.25}$$

where we have also taken into account the contributions from the undeformed vertex with coupling λ. Sticking for a moment to the interpretation of T as temperature, this is nothing but the thermal mass of the scalar fields.

The figure shows the behaviour of the non-planar amplitude as a function of b in $d = 3$. The external momentum corresponds to the mode numbers $n_p = 3$ and $n_p = 10$. The higher the mode number is, the faster is the oscillation in b. For $b \to 0$ the amplitude grows without bounds. However, at precisely $b = 0$ the zeta-function regularization sets in and there the amplitude is finite! The masses of the KK-states are given by the planar and non-planar contributions

$$M^2_{\rm KK} = 4\pi^2 n_p^2 T^2 + \mathcal{A}_{\rm p} + \mathcal{A}_{\rm n-p} =$$

$$= 4\pi^2 n_p^2 T^2 + (2\lambda + g)\frac{T^2}{6} + g\frac{T^2}{2\pi^2 b^2} \ , \tag{2.26}$$

where in the last line we made an expansion for small b, keeping only the leading term in $d = 3$ and without twist. For small b the correction to the KK-mass is independent from the mode number. Since b can be arbitrarily small, the non-planar part can give rise to very large KK-masses. Note, however, that in the case of a commutator interaction at tree level, $g = -\lambda$, the contribution of the non-planar amplitude is negative! The KK-mass is then reduced by the non-planar contribution. In fact, for sufficiently small b the value of M^2_{KK} can become negative for the lowest mode numbers! This presumably means that the corresponding field in the d-dimensional theory becomes tachyonic and develops a vacuum expectation value!

For $D = 4$ ($d = 3$), we have seen in our discussion of the non-compact case that dipole interaction terms with multiples of the dipole length L are necessary. These interactions give rise to non-planar correction to the KK-masses of the same form but with b replaced by the reduction b_k to $(-1/2, 1/2]$ of kTL. It can occur then that, although b is not small, b_k is small and could lead to large contributions. We argued that it is consistent to assume that these interactions are of order g^k but this might not be enough to suppress the non-planar amplitude, as happens for example by taking $TL = b = 0.333$, $b_3 = -0.001$ and $g = 0.1$ which gives $g^3/b_3^2 \sim 10^3$!

3. The 'Wess–Zumino' dipole theory

In this section we will do the same analysis as in the preceding subsection, but now for a theory with bosons and fermions. We shall start thus by considering the Wess–Zumino action in $D = 4$.

We need a U(1) symmetry to define the dipole deformation and we will chose the $\mathrm{U}(1)_{\mathcal{R}}$ of the supersymmetry algebra. Therefore, we must consider the massless Wess–Zumino theory. We shall drop also one star-product inside the integral, so that the propagators of this theory are the same as those in the commutative one. The action is

$$\begin{aligned} S_{\star\mathrm{WZ}} = \int \mathrm{d}^D x \Big\{ & \partial_\mu \phi^\dagger \partial^\mu \phi - i\psi\sigma^\mu \partial_\mu \overline{\psi} - g^2\, \phi^\dagger \star \phi^\dagger \star \phi \star \phi - \\ & -g\, \phi \star \psi^\alpha \star \psi_\alpha - g\, \phi^\dagger \star \overline{\psi}_{\dot\alpha} \star \overline{\psi}^{\dot\alpha} \Big\} \ . \end{aligned} \tag{3.1}$$

It is interesting to see explicitly how supersymmetry is broken by the dipole deformation. To do so, we go back to the action with the auxiliary fields

$$S_{\star\mathrm{WZ}} = \int \mathrm{d}^D x \left(\partial_\mu \phi^\dagger \partial^\mu \phi - i\psi\sigma^\mu \partial_\mu \overline{\psi} + F^\dagger F + g\Big(\phi \star \phi \star F - \ \phi \star \psi^\alpha \star \psi_\alpha \Big) + \mathrm{h.c.} \right) \ , \tag{3.2}$$

where we see that the correct dipole vectors are $L_\phi = -2L_\psi$ and $L_F = -2L_\phi$. Note that upon integrating out the auxiliary fields F, $F^\dagger$ we only generate the quartic scalar field coupling $\phi^\dagger \star \phi^\dagger \star \phi \star \phi$! Plugging in it the following susy variations

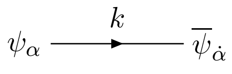
$$\delta_\xi\, \phi = \sqrt{2}\, \xi\psi \ , \quad \delta_\xi\, \psi_\alpha = i\sqrt{2}\, (\sigma^\mu\, \overline{\xi})_\alpha\, \partial_\mu \phi + \sqrt{2}\, \xi_\alpha F \ , \quad \delta_\xi\, F = i\sqrt{2}\, (\overline{\xi}\, \overline{\sigma}^\mu)^\alpha\, \partial_\mu \psi_\alpha \ , \tag{3.3}$$

one obtains

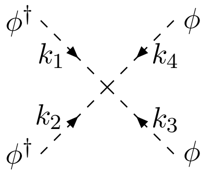
$$\delta_\xi S_{\star \mathrm{WZ}} = \int \mathrm{d}^D x \Big\{ \delta_\xi \,(\text{kinetic}) + g\, \delta_\xi\, \phi \star \phi \star F + g\, \phi \star \delta_\xi\, \phi \star F + g\, \phi \star \phi \star\, \delta_\xi\, F \; - \\ - g\, \delta_\xi\, \phi \star \psi^\alpha \star \psi_\alpha - g\, \phi \star \delta_\xi\, \psi^\alpha \star \psi_\alpha - g\, \phi \star \psi^\alpha \star \delta_\xi\, \psi_\alpha \Big\} \,. \tag{3.4}$$

It is clear that none of these terms satisfy $\sum L_i = 0$, because the $\mathcal{R}$-symmetry does not commute with supersymmetry. As a consequence, one cannot delete one star-product nor use the cyclicity condition in (3.4). This simply means that the U(1) used for the dipole deformation cannot be regarded as the $\mathrm{U}(1)_{\mathcal{R}}$-symmetry of a supersymmetric theory: our action is not the non-commutative Wess–Zumino action, but we will keep this name as it still describes a theory of bosons and fermions.

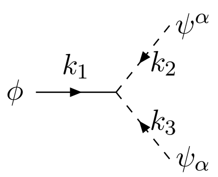
The Feynman rules for this theory are



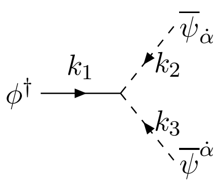
$$= \frac{i\,\sigma^\mu_{\alpha\dot\alpha}\, \boldsymbol{k}_\mu}{\boldsymbol{k}^2 - m^2 + i\varepsilon} \,, \tag{3.5}$$



$$= -ig^2 \, \exp\left(-\frac{i}{2} L_\phi \cdot (\boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3 + \boldsymbol{k}_4) \right) \,, \tag{3.6}$$



$$= -ig \, \exp\left(\frac{i}{2} L_\psi \cdot (\boldsymbol{k}_2 - \boldsymbol{k}_3) \right) \,, \tag{3.7}$$



$$= -ig \, \exp\left(-\frac{i}{2} L_\psi \cdot (\boldsymbol{k}_2 - \boldsymbol{k}_3) \right) \quad \text{since } L_{\overline{\psi}} = -L_\psi \,. \tag{3.8}$$

Our aim is to see how the breaking of supersymmetry manifests itself in the non-planar sector of the theory. As it is well-known, supersymmetry guaranties the cancellation of quadratic divergences and we will therefore study these and the corresponding non-planar amplitudes that are regulated by the dipole length. We will limit ourselves therefore to investigate the one-loop corrections to the scalar two-point function.

3.1 1-loop non-planar correction to bosonic propagator

We consider as before the theory on $\mathcal{M} = \mathbb{R}^{1,D-2} \times \mathbf{S}^1$, and also allow for twisted boundary conditions on the fields

$$\phi(x + 2\pi R) = \mathrm{e}^{2\pi i \alpha}\, \phi(x) \,, \tag{3.9}$$

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$$\psi(x + 2\pi R) = \mathrm{e}^{-\pi i\alpha}\,\psi(x)\ , \tag{3.10}$$

where $\alpha = 0$ are supersymmetric boundary conditions whereas $\alpha = 1$ corresponds to the boundary conditions of thermal field theory. We will take the dipole vectors to be along the $\mathbf{S}^1$ again.

The bosonic part can be written directly copying from (2.21), noticing that $b = -2b_\psi$, and changing appropriately the coupling ($g \to g^2$). On the other hand, taking into account the combinatorics with a minus sign from the fermion loop and a $\frac{1}{2}$ from the symmetry factor, the non-planar 1-loop fermionic amplitude is

$$-i\mathcal{A}^{\mathrm{F}}_{\mathrm{n-p}} = -2g^2 \int \frac{\mathrm{d}^D \boldsymbol{k}}{(2\pi)^D}\ \cos\left(\frac{L_\psi}{R}\,(n_p + 2n)\right) \frac{(\boldsymbol{p}+\boldsymbol{k})\cdot\boldsymbol{k}}{(\boldsymbol{p}+\boldsymbol{k})^2\,\boldsymbol{k}^2}\ , \tag{3.11}$$

where again n_p and n are the external and loop momenta along the $\mathbf{S}^1$. Also notice that the twist parameter α drops out from the argument of the cosine. Besides, we have introduced $b_\psi \equiv TL_\psi$ reduced to its non-integer part, $b_\psi \in \left[-\frac{1}{2}, \frac{1}{2}\right)$, because of the periodicity of the cosine function.

We will make an expansion in $\boldsymbol{p}$, and compute the leading and subleading contributions,

$$\frac{(\boldsymbol{p}+\boldsymbol{k})\cdot\boldsymbol{k}}{(\boldsymbol{p}+\boldsymbol{k})^2\,\boldsymbol{k}^2} = \frac{1}{\boldsymbol{k}^2} - \frac{\boldsymbol{p}\cdot\boldsymbol{k}}{\boldsymbol{k}^4} + \left(2\frac{(\boldsymbol{p}\cdot\boldsymbol{k})^2}{\boldsymbol{k}^6} - \frac{\boldsymbol{p}^2}{\boldsymbol{k}^4}\right) + \mathcal{O}(\boldsymbol{p}^3)\ , \tag{3.12}$$

which includes all the superficially divergent terms (since $D = 4$ is implied): quadratic + linear + logarithmic. We will evaluate the corresponding non-planar contributions. After a Wick rotation of the theory, we give next the expression for each term.

Quadratic part: The part of the non-planar amplitude corresponding to a quadratic divergence is

$$\mathcal{A}^{\mathrm{F},(2)}_{\mathrm{n-p}} = -2g^2 T \int \frac{\mathrm{d}^d k_E}{(2\pi)^d} \sum_{n\in\mathbb{Z}}\ \cos\Big(2\pi b_\psi\,(n_p + 2n)\Big)\ \frac{1}{k_E^2 + 4\pi^2 T^2 (n-\alpha/2)^2}\ , \tag{3.13}$$

and is the same as (2.21) with the substitutions $g \to g^2$, $\alpha \to \alpha/2$ and $n_p \to n_p/2$. So the part of the combined amplitude corresponding to the quadratic divergences in $D = 4$ is

$$\mathcal{A}^{(2)}_{\mathrm{n-p}} = g^2 \frac{T^2}{2\pi^2}\,\mathrm{Re}\left[\mathrm{e}^{2i\pi b_\phi (n_p+\alpha)} Z \begin{vmatrix} b_\phi \\ \alpha \end{vmatrix} (d-1) - \mathrm{e}^{-2i\pi b_\psi (n_p+\alpha)} Z \begin{vmatrix} b_\phi \\ \alpha/2 \end{vmatrix} (d-1)\right]\ . \tag{3.14}$$

The planar contributions are obtained by setting $b_\phi = b_\psi = 0$, and cancel in the case of supersymmetric boundary conditions $\alpha = 0$. The non-planar bosonic and fermionic contributions *do not cancel* each other even in the case of supersymmetric boundary conditions. This is a direct consequence of the fact that the amplitudes depend explicitly on the different dipole lengths related to the $\mathcal{R}$-charges through b_ϕ and b_ψ respectively.

Linear part: We will assume that the d-dimensional inflowing momentum p vanishes, since we want to compute the amputated correction to the two-point function. Therefore $\boldsymbol{p} = (n_p + \alpha)/R$. The part of the non-planar amplitude giving the linear divergence thus is

$$\mathcal{A}^{\mathrm{F},(1)}_{\mathrm{n-p}} = 2g^2 T \int \frac{\mathrm{d}^d k_E}{(2\pi)^d} \sum_{n\in\mathbb{Z}}\ \cos\Big(2\pi b_\psi\,(n_p + 2n)\Big)\ \frac{(n_p+\alpha)(n-\alpha/2)/R^2}{\Big[k_E^2 + 4\pi^2 T^2 (n-\alpha/2)^2\Big]^2}\ . \tag{3.15}$$

In $D=4$, the final expression for this divergence is

$$\mathcal{A}^{\mathrm{F},(1)}_{\mathrm{n-p}} = -2g^2T^2\,(n_p+\alpha)\ \mathrm{Re}\left[\mathrm{e}^{-\pi i b_\phi n_p}\left(\frac{1}{2\pi i}\frac{\partial}{\partial b_\phi}+\frac{\alpha}{2}\right) Z\begin{vmatrix}\alpha/2\\ b_\phi\end{vmatrix}(4-d)\right]\,, \tag{3.16}$$

where all the computations are contained in appendix B. The derivative of the Epstein zeta function must be understood for $b_\phi \neq 0$: one cannot get the result for $b_\phi = 0$ as the limit $b_\phi \to 0$ in this expression. In such a case, one simply has to trace back to the series and see that it is an alternating sum. Therefore, in that case the linear divergence vanishes.

Logarithmic part: The last two terms in the power expansion (3.12) give the logarithmically divergent contributions. As before, we assume that the external momentum only runs along the circle. After Wick rotation, these terms look like

$$\begin{aligned}\mathcal{A}^{\mathrm{F},(0)}_{\mathrm{n-p}} &= -2g^2T\int\frac{\mathrm{d}^dk_E}{(2\pi)^d}\sum_{n\in\mathbb{Z}}\ \cos\Big(2\pi b_\psi\,(n_p+2n)\Big)\times \\ &\qquad\times\left\{2\,\frac{(n_p+\alpha)^2(n-\alpha/2)^2/R^4}{\Big[k_E^2+4\pi^2T^2(n-\alpha/2)^2\Big]^3}-\frac{(n_p+\alpha)^2/R^2}{\Big[k_E^2+4\pi^2T^2(n-\alpha/2)^2\Big]^2}\right\}\,,\end{aligned} \tag{3.17}$$

which after all computations in appendix B, and specializing to $D=4$, is

$$\mathcal{A}^{\mathrm{F},(0)}_{\mathrm{n-p}} = \frac{g^2T^2}{4}\,(n_p+\alpha)^2\ \mathrm{Re}\left[\mathrm{e}^{-\pi i b_\phi n_p}\ Z\begin{vmatrix}\alpha/2\\ b_\phi\end{vmatrix}(4-d)\right]\,. \tag{3.18}$$

4. Gauge dipole field theory

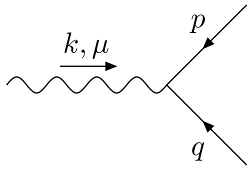
Finally we want to have a brief look to the dipole deformation of QED. Until now all the U(1) symmetries that we used for the dipole deformations were global symmetries. Now we want to use the local U(1) symmetry. Since the gauge field is neutral it has dipole length zero. The field strength tensor is therefore undeformed. The dipole deformation allows for three different actions of the gauge group:

- left action matter fields
  $\psi \to U\star\psi$: $\quad D_\mu\psi = \partial_\mu + igA_\mu\star\psi = \partial_\mu + igA_\mu(x-\frac{L}{2})\psi(x)$
- right action matter fields
  $\psi \to \psi\star U^\dagger$: $\quad D_\mu\psi = \partial_\mu - ig\psi\star A_\mu = \partial_\mu - igA_\mu(x+\frac{L}{2})\psi(x)$
- adjoint action matter fields
  $\psi \to U\star\psi\star U^\dagger$: $D_\mu\psi = \partial_\mu + ig[\psi, A_\mu]_\star = \partial_\mu + ig\Big(A_\mu(x-\frac{L}{2})\psi(x) - \psi(x)A_\mu(x+\frac{L}{2})\Big)$

Let us choose a commutator-like interaction of the gauge field with the Dirac spinor Ψ. The action is

$$S_{\star\mathrm{QED}} = \int\mathrm{d}^4x\,\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar\Psi i\gamma^\mu(\partial_\mu + igA_\mu\star\Psi - ig\Psi\star A_\mu) - m\bar\Psi\Psi\right)\,. \tag{4.1}$$

The Feyman rule for the interaction vertex is



$$\sim \; 2ig\gamma^\mu \sin\left(\frac{k\cdot L}{2}\right) .$$

The polarization tensor for the gauge field at one loop is

$$\Pi^{\mu\nu} = 4g^2 \sin\left(\frac{p\cdot L}{2}\right)^2 \int \frac{\mathrm{d}^4k}{(2\pi)^4} \;\mathrm{tr}\; \left[\gamma^\mu \frac{1}{\not{k} - m} \gamma^\nu \frac{1}{\not{k} - \not{p} - m}\right] . \tag{4.2}$$

We see that the Feynman integral is unchanged compared to the usual QED. However, the amplitude does crucially depend on the inflowing momentum through the $\sin^2$ term! This means that the logarithmic divergence of the integral is multiplied with $\sin^2(pL/2)$. It is important to realize that this happens here for a two-point function. We argued however that the star-product necessarily must have zero effect on the tree-level two-point functions. Therefore, it is impossible to absorb the appearing logarithmic divergence in the fields or parameters of the tree level Lagrangian! The theory is not renormalizable, and contrary to what we found in the scalar interaction case, there is no star-product term that could be introduced to deal with this divergence! The problem can be traced back to the neutrality of the gauge field under the U(1) symmetry and the commutator interaction. If instead we use left (or right) multiplication the phases of the star-product just cancel in the one-loop polarization tensor.

5. Conclusions

We have reinvestigated certain aspects of dipole deformed field theories. Our emphasis was on the computation of quantum corrections to the KK-state masses in a simple compactification. Along the way we also found that dipole scalar field theory might allow for spontaneous breaking of translation symmetry. Our analysis was based on a simple one-loop computation and it is not clear if the properties of the dispersion relation allowing for this phase transition persist to higher loops or non-perturbative corrections. However, this is an addressable problem. Lattice studies in the case of the Moyal bracket deformed theory have shown the formation of stripe phases. It should not be too difficult to formulate dipole deformed theories on the lattice and perform an analogous study.

The corrections to the masses of KK-states showed a very interesting pattern. The dipole length L together with the radius of compactification $R = 1/(2\pi T)$, forms a dimensionless parameter which we called b. It is remarkable that this parameter is compact, i.e. takes values only in the interval $(-1/2, 1/2]$. The interesting corrections stem from non-planar graphs, in which the UV-divergences are regulated by the presence b. For $b \to 0$ the regularization becomes less effective, and therefore the non-planar contribution becomes very large and can even overwhelm the tree-level contribution. Depending on the

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form of the tree level interaction it might be the case that the non-planar graph contributes with a minus sign to the square of the KK-mass and for small enough b the corresponding mode might even become tachyonic!

We also have seen that in the dipole deformation of a supersymmetric theory in which the $\mathrm{U}(1)_{\mathcal{R}}$ symmetry is used for the deformation the supersymmetry is broken. In the planar sector the quadratic divergences still cancel but in the non-planar sector the contributions corresponding to quadratic divergences at $b = 0$ do not cancel due to the different dipole lengths of the fields circulating the loop.

Finally, we showed that dipole-deformed QED with adjoint action of the gauge group is not renormalizable in a way that would only allow star-product terms in the tree level Lagrangian. This problem might be cured only in a highly supersymmetric extension like the one based on the $\mathcal{N} = 4$ theory.

Acknowledgments

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A. Epstein zeta function

All of the 1-loop calculations that we perform in the compact case can be given a closed expression in terms of Epstein zeta functions, a zeta function for quadratic forms. For a positive integer p, let $\vec{g}, \vec{h} \in \mathbb{R}^p$, $\vec{n} \in \mathbb{Z}^p$. Let us further define the scalar product of any two vectors in $\mathbb{R}^p$ as $(\vec{g}, \vec{h}) = \sum_{i=1}^{p} g_i\, h_i$, and the positive definite quadratic form as

$$\varphi(\vec{x}) = \sum_{i,j=1}^{p} c_{ij} x_i x_j \ , \tag{A.1}$$

where (c_{ij}) is called the *module*. The Epstein zeta function of order p and *characteristic* $\begin{vmatrix} \vec{g} \\ \vec{h} \end{vmatrix}$ is defined as a function of the complex variable s as [18]

$$Z \begin{vmatrix} \vec{g} \\ \vec{h} \end{vmatrix} (s)_\varphi := \sum_{\vec{n} \in \mathbb{Z}^p} {}' [\varphi(\vec{n} + \vec{g})]^{-s/2} \ \mathrm{e}^{2\pi i (\vec{n}, \vec{h})} \ . \tag{A.2}$$

Notice the prime in the summation, indicating that in case $\vec{g}$ belongs to the integer lattice one has to subtract the value of $\vec{n}$ such that $\vec{n} + \vec{g}$ vanishes.

We will focus on $p = 1$ with module the identity. Thus $[\varphi(n+g)]^{-s/2} = |n+g|^{-s}$; hence

$$Z\begin{vmatrix} g \\ h \end{vmatrix}(s) = \sum_{n\in\mathbb{Z}}{}' \frac{\mathrm{e}^{2\pi inh}}{|n+g|^s} \ . \tag{A.3}$$

Our aim is to give the analytic continuation of this zeta function over the complex plane. To extract the singular structure one first obtains the integral representation. Using the Euler Gamma function, $\Gamma(\alpha) = \int_0^\infty \mathrm{d}t\, t^{\alpha-1}\,\mathrm{e}^{-t}$; with a change of variables $\alpha = \frac{s}{2}$, $t = \pi z^2\,\xi$ and $z = |n+g|$, one arrives at

$$Z\begin{vmatrix} g \\ h \end{vmatrix}(s) = \frac{\pi^{s/2}}{\Gamma(s/2)} \sum_{n\in\mathbb{Z}}{}' \int_0^\infty \mathrm{d}\xi\ \xi^{(s-2)/2}\ \mathrm{e}^{-\pi\xi(n+g)^2+2\pi inh} \ . \tag{A.4}$$

Now one splits the integration interval as $\int_0^\infty = \int_0^1 + \int_1^\infty$. In the first integral perform a change of variables $\xi = t^{-1}$ and a Poisson resummation

$$\sum_{n=-\infty}^{\infty} \mathrm{e}^{-\pi n^2\tau+2n\pi z\tau} = \frac{\mathrm{e}^{\pi\tau z^2}}{\sqrt{\tau}} \sum_{m=-\infty}^{\infty} \mathrm{e}^{-\pi m^2/\tau-2\pi imz} \ . \tag{A.5}$$

Recalling the definition of the Jacobi ϑ_3-function

$$\vartheta_3(z|\tau) := \sum_{n\in\mathbb{Z}} \mathrm{e}^{i\pi n^2\tau+2\pi inz} \ , \tag{A.6}$$

we have the following expression for the Epstein zeta function

$$\begin{aligned} Z\begin{vmatrix} g \\ h \end{vmatrix}(s) = \frac{\pi^{s/2}}{\Gamma(s/2)} \int_1^\infty \mathrm{d}t \Big\{ & t^{(s-2)/2}\ \mathrm{e}^{-\pi g^2 t}\ \vartheta_3(h+igt|it) + \\ & + t^{-(s+1)/2}\ \mathrm{e}^{-\pi h^2 t-2\pi igh}\ \vartheta_3(g-iht|it) \Big\} \ . \end{aligned} \tag{A.7}$$

Singularities appear in each term for $n = 0$ and either $g = 0$ or $h = 0$. Adding and subtracting those terms from the summations, and integrating formally yields the integral representation of the Epstein zeta function

$$\begin{aligned} Z\begin{vmatrix} g \\ h \end{vmatrix}(s) = \frac{\pi^{s/2}}{\Gamma(s/2)} \Bigg\{ & \frac{2\delta_{h,0}}{s-1} - \frac{2\delta_{g,0}}{s} + \\ & + \int_1^\infty \mathrm{d}t \Big[t^{(s-2)/2}\ \mathrm{e}^{-\pi g^2 t} \Big(\vartheta_3(h+igt|it) - \delta_{g,0}\Big) + \\ & + t^{-(s+1)/2}\ \mathrm{e}^{-\pi h^2 t-2\pi igh} \Big(\vartheta_3(g-iht|it) - \delta_{h,0}\Big)\Big]\Bigg\} \ . \end{aligned} \tag{A.8}$$

Now it is clear that the function is meromorphic in the whole complex s-plane with a simple pole at $s = 1$.

Functional identity. Besides the straightforward relation

$$Z\left|\begin{matrix}-g\\-h\end{matrix}\right|(s)=Z\left|\begin{matrix}g\\h\end{matrix}\right|(s)\;, \tag{A.9}$$

one can obtain the functional identity for the Epstein zeta function, which interchanges the values of g and h. Starting with the integral representation (A.4), and after a Poisson resummation and a polar change of variables $t\to t^{-1}$,

$$\frac{\Gamma\left(\frac{s}{2}\right)}{\pi^{s/2}}\;Z\left|\begin{matrix}g\\h\end{matrix}\right|(s)=\mathrm{e}^{-2\pi igh}\;\frac{\Gamma\left(\frac{1-s}{2}\right)}{\pi^{(1-s)/2}}\;Z\left|\begin{matrix}h\\-g\end{matrix}\right|(1-s)\;. \tag{A.10}$$

B. Explicit computation for dipole WZ

This appendix shows the calculations performed to get the expressions for the linear divergence (3.16), and the logarithmic divergence (3.18).

As said in the text, taking the inflowing momentum only along the circumference gives the following linearly divergent term

$$\mathcal{A}^{\mathrm{F},(1)}_{\mathrm{n-p}}=2g^2T\int\frac{\mathrm{d}^dk_E}{(2\pi)^d}\sum_{n\in\mathbb{Z}}\;\cos\Big(2\pi b_\psi\left(n_p+2n\right)\Big)\;\frac{(n_p+\alpha)(n-\alpha/2)/R^2}{\Big[k_E^2+4\pi^2T^2(n-\alpha/2)^2\Big]^2}\;. \tag{B.1}$$

One can put the cosine function into exponentials, and use the Schwinger parametrization in the denominator to perform a Gaussian integration. Then

$$\begin{aligned}\mathcal{A}^{\mathrm{F},(1)}_{\mathrm{n-p}}\;&=4\pi^2g^2T^3\left(n_p+\alpha\right)\sum_{n\in\mathbb{Z}}\Bigg\{\left(n-\frac{\alpha}{2}\right)\left[\mathrm{e}^{2\pi ib_\psi n_p}\mathrm{e}^{4\pi ib_\psi n}+\text{c.c.}\right]\times\\&\qquad\times\int_0^\infty\mathrm{d}\xi\;\xi\int\frac{\mathrm{d}^dk_E}{(2\pi)^d}\mathrm{e}^{-\xi k_E^2-4\pi^2T^2\xi(n-\alpha/2)^2}\Bigg\}\;.\end{aligned} \tag{B.2}$$

Performing the Gaussian integration, and further doing $t=4\pi T^2\xi$ yields

$$\begin{aligned}\mathcal{A}^{\mathrm{F},(1)}_{\mathrm{n-p}}\;&=g^2T^{d-1}\left(n_p+\alpha\right)\sum_{n\in\mathbb{Z}}\Bigg\{\left(n-\frac{\alpha}{2}\right)\left[\mathrm{e}^{2\pi ib_\psi n_p}\mathrm{e}^{4\pi ib_\psi n}+\text{c.c.}\right]\times\\&\qquad\times\int_0^\infty\mathrm{d}t\;t^{(2-d)/2}\mathrm{e}^{-\pi t(n-\alpha/2)^2}\Bigg\}\;.\end{aligned} \tag{B.3}$$

Using the Euler Gamma function as we did in appendix A, and recalling $b_\phi=-2b_\psi$, one arrives at

$$\mathcal{A}^{\mathrm{F},(1)}_{\mathrm{n-p}}=2g^2T^{d-1}\left(n_p+\alpha\right)\;\frac{\Gamma\left(2-\frac{d}{2}\right)}{\pi^{(4-d)/2}}\,\mathrm{Re}\left[\mathrm{e}^{-\pi ib_\phi n_p}\sum_{n\in\mathbb{Z}}\left(n-\frac{\alpha}{2}\right)\frac{\mathrm{e}^{-2\pi ib_\phi n}}{|n-\alpha/2|^{4-d}}\right]\;. \tag{B.4}$$

This expression can be given formally in terms of the Epstein zeta function and its derivative with respect to b

$$\mathcal{A}^{\mathrm{F},(1)}_{\mathrm{n-p}}=-2g^2T^{d-1}\left(n_p+\alpha\right)\;\frac{\Gamma\left(2-\frac{d}{2}\right)}{\pi^{(4-d)/2}}\,\mathrm{Re}\left[\mathrm{e}^{-\pi ib_\phi n_p}\left(\frac{1}{2\pi i}\frac{\partial}{\partial b_\phi}+\frac{\alpha}{2}\right)Z\left|\begin{matrix}\alpha/2\\b_\phi\end{matrix}\right|(4-d)\right]\;. \tag{B.5}$$

Now we do the computation for the logarithmically divergent terms. As before, we can put the cosine as exponentials in (3.17), and use the Schwinger trick to get, after Gaussian integration

$$\begin{aligned}\mathcal{A}^{\mathrm{F},(0)}_{\mathrm{n-p}} &= -\frac{4\pi^2 g^2 T^3}{(2\sqrt{\pi})^d}\,(n_p+\alpha)^2\Bigg\{\left(\mathrm{e}^{-\pi i b_\phi (n_p+2n)} + \mathrm{c.c.}\right)\times \\ &\qquad \times\int_0^\infty \mathrm{d}\xi\; \xi^{-d/2}\left(4\pi^2T^2\xi^2(n-\alpha/2)^2-\xi\right)\,\mathrm{e}^{-4\pi^2T^2\xi(n-\alpha/2)^2}\Bigg\}\,. \qquad \text{(B.6)}\end{aligned}$$

Performing the change of variables $t = 4\pi T^2\xi$, and mapping to the Euler Gamma function as with the linear divergence, one obtains

$$\mathcal{A}^{\mathrm{F},(0)}_{\mathrm{n-p}} = -\frac{g^2T^{d-1}}{4}\,(n_p+\alpha)^2\left(1-\frac{d}{2}\right)\frac{\Gamma\left(2-\frac{d}{2}\right)}{\pi^{(4-d)/2}}\sum_{n\in\mathbb{Z}}\left(\mathrm{e}^{-\pi i b_\phi(n_p+2n)}+\mathrm{c.c.}\right)\frac{1}{|n-\alpha/2|^{4-d}}\,, \qquad \text{(B.7)}$$

and after some algebra it can be given in terms of the Epstein zeta function

$$\mathcal{A}^{\mathrm{F},(0)}_{\mathrm{n-p}} = -\frac{g^2T^{d-1}}{4}\,(n_p+\alpha)^2\left(1-\frac{d}{2}\right)\frac{\Gamma\left(2-\frac{d}{2}\right)}{\pi^{(4-d)/2}}\,\mathrm{Re}\left[\mathrm{e}^{-\pi i b_\phi n_p} Z\begin{vmatrix}\alpha/2\\ b_\phi\end{vmatrix}(4-d)\right]\,. \qquad \text{(B.8)}$$

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